647510 **MSC2MatC201x** **Seat No :________**

M.Sc.(Maths.) Semester - 2 (*CBCS*) Examination
March/April - 2024 (NEW COURSE)
Algebra 2 (Core)

Time: 02:30 Hours **Marks: 70**

Instructions:

1. All questions are compulsory.
2. Figures to the right indicate marks.

Q.1 (A) Answer any TWO of the following questions. **[10]**

(1) Define Primitive Polynomial. Let $f(x) \in Z[x]$ be primitive. Then $f(x)$ is reducible over Q if and only if $f(x)$ is reducible over Z.

(2) State and prove Eisenstein Criterion.

(3) Let $f(x) \in F[x]$. Then $f(x)$ has multiple roots if and only if $f(x)$ and $f'(x)$ have nontrivial common factor.

Q.1 (B) Answer any ONE of the following questions. **[04]**

(1) Find the splitting field of the polynomial $p(x) = (x-1)^2(x^3-2) \in Q[x]$.

(2) Show that a polynomial of degree n over a field can have at most n roots in any extension field.

Q.2 (A) Answer any TWO of the following questions. **[10]**

(1) Let $p(x)$ be an irreducible polynomial in $F[x]$. Then there exists an extension E of F in which $p(x)$ has a root.

(2) If K is a finite extension of F, then $G(K,F)$ is a finite group with $o\big(G(K,f)\big) \leq [K:f]$.

(3) Let E be an algebraic extension of F, and $\sigma: E \to E$ be an embedding of E into itself over F. Then σ is onto and, hence, an automorphism of E.

Q.2 (B) Answer any ONE of the following questions. **[04]**

(1) Compute $G(\mathbb{C}:\mathbb{R})$.

(2) State and prove Fundamental Theorem of Galios Theory.

Q.3 (A) Answer any TWO of the following questions. **[10]**

(1) Let F be a field. Then show that there exists an extension $\bar{F}$ that is algebraic over F and is algebraically closed; that is, each field has an algebraic closure.

(2) Define Module and give two examples of it.

(3) If M is an R-module and $x \in M$, then show that the set $K = \{rx + nx | r \in R, n \in Z\}$ is an R-submodule of M containing x.

Q.3 (B) Answer any ONE of the following questions. **[04]**

(1) If an R-module M is generated by a set $\{x_1, x_2, \dots, x_m\}$ and $1 \in R$, then $M = \{r_1x_1 + r_2x_2 + \cdots + r_nx_n | r_i \in R\}$.

(2) Define Submodule of a module of a module with an example. Show that the polynomial ring $R[x]$ over a ring R is an R −module.

Q.4 (A) Answer any TWO of the following questions.

(1) State and prove Fundamental Theorem of R-homomorphism.

(2) Define Cyclic Module. Let R be a ring with unity. Prove that an R −module M is cyclic if and only if $M \simeq R/I$ for some left ideal I of R.

(3) The submodules of a quotient module M/N are of the form U/N, where U is a submodule of M containing N.

Q.4 (B) Answer any ONE of the following questions.

(1) State and prove Schur's Lemma.

(2) Let M be a finitely generated free module over a commutative ring R. Then show that all the bases of M have the same number of elements.

Q.5 (A) Answer any TWO of the following questions. **[10]**

(1) Let $f: M \to N$ is an $R-$homomorphiism of an $R-$module M to an $R-$module N. Show that:

i. $Ker\ f$ is an $R-$submodule of M.

ii. $Im\ f$ is an $R-$submodule of N.

(2) Let $F \subset E$ and $f(x), g(x) \in F[x]$. If $f(x)$ and $g(x)$ have a nontrivial common factor in $E[x]$, then prove that they have a nontrivial common factor in $F[x]$.

(3) Let M be a finitely generated free module over a commutative ring R. Then show that all the basis of M have the same number of elements.

Q.5 (B) Answer any ONE of the following questions. **[04]**

(1) Let R be a ring with unity. An R-module M is cyclic if and only if $M \cong R/I$, for some left ideal I of R.

(2) Define the following terms:

i. Finitely Generated Module

ii. $R-$homomorphism of modules

iii. Completely Reducible Module

iv. Rank of a Module

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